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List-edge and list-total colorings of graphs embedded on hyperbolic surfaces

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Abstract

In the paper, we prove that if G is a graph embeddable on a surface of Euler characteristic $\varepsilon < 0$ and $\Delta \geq \sqrt{25-24\varepsilon}+10$, then $\chi'_{\mathrm{list}}(G) = \Delta$ and $\chi''_{\mathrm{list}}(G) = \Delta + 1$. This extends a result of Borodin, Kostochka and Woodall [O.V. Borodin, A.V. Kostochka, D.R. Woodall, List-edge and list-total colorings of multigraphs, J. Comb. Theory Series B 71 (1997) 184–204].

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1. Introduction

In this paper, all graphs are finite, simple and undirected. For a graph G, we use $V(G)$, $E(G)$ and Δ to denote the vertex set, the edge set and the maximum degree of G. Let $VE(G) = V(G) \cup E(G)$. The set of neighbors of v is denoted by $N(v)$ for $v \in V(G)$ and the degree of the face f, that is, the number of edges around f, is denoted by $r(f)$. A *k-vertex* is a vertex of degree k. A *k-face* is a face incident with k edges unless they are cut edges in that case each cut edge is counted twice. A *proper total coloring* of a graph G is a coloring of $VE(G)$ such that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi''(G)$ is the smallest number of colors such that G has a proper total coloring. A graph G is said to be *totally f-choosable* if, whenever we give a list A_x of $f(x)$ colors to each element $x \in VE(G)$, there exists a proper total coloring of G where each element is colored with a color from its own list. If $f(x) = k$ for every element $x \in VE(G)$, we say G is *totally k-choosable*. The *list-total chromatic number* $\chi''_{\mathrm{list}}(G)$ is the smallest integer k such that G is totally k-choosable. The *list-edge chromatic number* $\chi'_{\mathrm{list}}(G)$ of G is defined similarly in terms of coloring edges alone, as well as the concept of *edge f-choosable*. The ordinary edge chromatic number is denoted by $\chi'(G)$. Obviously, $\chi'_{\mathrm{list}}(G) \geq \chi'(G) \geq \Delta(G)$ and $\chi''_{\mathrm{list}}(G) \geq \chi''(G) \geq \Delta + 1$.

Conjecture. *For any graph G,* (a) $\chi'_{\mathrm{list}}(G) = \chi'(G)$ *and* (b) $\chi''_{\mathrm{list}}(G) = \chi''(G)$.

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Table 1
Results on graph G embedded in a surface of Euler characteristic $-5 \leq \varepsilon \leq -1$

ε	Heawood number [3] $H(\varepsilon) = \left\lfloor \frac{7+\sqrt{49-24\varepsilon}}{2} \right\rfloor$	Zhao's result [9,4] $\chi'(G) = \Delta$ if	Mel'nikov's result [6] $\chi'(G) = \Delta$ if	Zhao's result [10] $\chi''(G) \leq \Delta + 2$ if $\Delta \geq \left(\frac{20}{9}\right)(3-\varepsilon)+1$
−1	6	$\Delta \geq 9$	$\Delta \geq 10$	$\Delta \geq 10$
−2	7	$\Delta \geq 9$	$\Delta \geq 10$	$\Delta \geq 13$
−3	7	$\Delta \geq 10$	$\Delta \geq 11$	$\Delta \geq 15$
−4	8	$\Delta \geq 11$	$\Delta \geq 12$	$\Delta \geq 17$
−5	9	$\Delta \geq 11$	$\Delta \geq 12$	$\Delta \geq 19$
ε	Sanders' result [7] $\chi''(G) \leq \Delta + 2$ if $\Delta \geq 23 - 24\varepsilon$	Borodin's result [1] $\chi''_{\text{list}}(G) = \Delta + 1$ and $\chi'_{\text{list}}(G) = \Delta$ if $\Delta \geq \binom{t}{2} + 1$	Our result $\chi''_{\text{list}}(G) = \Delta + 1$ and $\chi'_{\text{list}}(G) = \Delta$ if $\Delta \geq 10 + \sqrt{25 - 24\varepsilon}$	
−1	$\Delta \geq 47$	$\Delta \geq 22$	$\Delta \geq 17$	
−2	$\Delta \geq 71$	$\Delta \geq 29$	$\Delta \geq 19$	
−3	$\Delta \geq 95$	$\Delta \geq 29$	$\Delta \geq 20$	
−4	$\Delta \geq 119$	$\Delta \geq 37$	$\Delta \geq 21$	
−5	$\Delta \geq 143$	$\Delta \geq 37$	$\Delta \geq 23$	

The part (a) of the conjecture was independently posed by Vizing, Gupta, Abertson and Collins, and Bollobás and Harris (see [5,8]), and is well known as the *the List Coloring Conjecture* or *the List-Edge-Coloring Conjecture, (LECC)*. Part (b) of the conjecture, also known as the List-Total-Coloring Conjecture (*LTCC*), was posed by Borodin et al. [1]. This conjecture is still very much open.

On the edge chromatic number, Mel'nikov [6] proved the following theorem in 1970, and in 1998, Hind and Zhao [4] and Yan and Zhao [9] improved Mel'nikov's result for $-5 \leq \varepsilon \leq 0$ (see Table 1). On the total chromatic number, Borodin, Kostochka and Woodall proved [2] in 1997 that $\chi''(G) = \Delta + 1$ if G is a planar graph with $\Delta \geq 11$. Zhao [10] showed that $\chi''(G) \leq \Delta + 2$ if $\Delta \geq (20/9)(3 - \varepsilon) + 1$ where $\varepsilon \leq -1$. In 2000, Sanders and Maharry [7] proved that $\chi''(G) \leq \Delta + 2$ if $\Delta \geq 23 - 24\varepsilon$ where $\varepsilon \leq 0$.

Theorem 1.1 (*[6]*). *If a graph G can be embedded on a surface of Euler characteristic $\varepsilon \leq 0$ and*

$$\Delta \geq \max\left\{ \left\lfloor \frac{11 + \sqrt{25 - 24\varepsilon}}{2} \right\rfloor, \left\lfloor \frac{8 + 2\sqrt{52 - 18\varepsilon}}{3} \right\rfloor \right\},$$

then $\chi'(G) = \Delta$.

For the graphs embedded on surface with nonnegative Euler characteristic, Borodin, Kostochka and Woodall [1] proved the following theorem in 1997. In the same paper, they also proved a similar result for graphs embedded on a surface with negative Euler characteristic and $\Delta \geq f(\varepsilon)$.

Theorem 1.2 (*[1]*). *Let $\Delta \geq 12$ and let G be a graph with maximum degree $\Delta(G) \leq \Delta$, embedded in a surface of nonnegative Euler characteristic. Then $\chi'_{\text{list}}(G) \leq \Delta$ and $\chi''_{\text{list}}(G) \leq \Delta + 1$. In particular, if $\Delta(G) = \Delta$, then $\chi'_{\text{list}}(G) = \Delta$ and $\chi''_{\text{list}}(G) = \Delta + 1$.*

Theorem 1.3 (*[1]*). *Let G be a simple graph with maximum degree Δ, embedded on a surface of Euler characteristic $\varepsilon \leq 0$ and $\Delta \geq \binom{t}{2} + 1$ where $t = \lceil 2 + \sqrt{16 - 6\varepsilon} \rceil$. Then $\chi'_{\text{list}}(G) = \Delta$ and $\chi''_{\text{list}}(G) = \Delta + 1$.*

In Section 3, we prove the following theorem.

Theorem 1.4. *If G is a graph with maximum degree Δ embedded on a surface of Euler characteristic $\varepsilon < 0$ and $\Delta \geq \sqrt{25 - 24\varepsilon} + 10$, then $\chi'_{\text{list}}(G) = \Delta$ and $\chi''_{\text{list}}(G) = \Delta + 1$.*

Since the lower bound on the maximum degree in Theorem 1.3 is linear in ε and our's involves $\varepsilon^{\frac{1}{2}}$, our bound on the maximum degree is better. Our lower bound on maximum degree is also better than Zhao's [10] and Sanders' [7] lower bound on maximum degree when $\varepsilon \leq -8$.

A critical edge Δ-choosable graph G is a graph with maximum degree Δ embedded on a surface of Euler characteristic $\varepsilon < 0$ such that G is not edge Δ-choosable and $G - x$ is edge Δ-choosable for any element $x \in VE(G)$. A critical totally $(\Delta + 1)$-choosable graph G is a graph with maximum degree Δ embedded on a surface of Euler characteristic $\varepsilon < 0$ such that G is not totally $(\Delta + 1)$-choosable and $G - x$ is totally $(\Delta + 1)$-choosable for any element $x \in VE(G)$. The concept of critical graph is often used in coloring problems, and, for example, the concept of total critical graph is used by Zhao [10]. A critical graph is also called a minimal counterexample in [1,2]. In the next section we will obtain structural information about critical graphs and show that certain configurations cannot occur in critical graphs.

2. Lemmas

This first lemma is for any bipartite graph. In fact, the following lemma is true for any bipartite multigraph.

Lemma 2.1 *([1]). A bipartite graph G is edge f-choosable where $f(uv) = \max\{d_G(u), d_G(v)\}$ for any $uv \in E(G)$.*

The next three lemmas are for critical graphs.

Lemma 2.2. *For any edge $uv \in E(G)$, if $\min\{d_G(u), d_G(v)\} \le \lfloor \frac{\Delta}{2} \rfloor$, then $d_G(u) + d_G(v) \ge \Delta + 2$.*

Proof. Let G be a critical edge Δ-choosable graph and let $uv \in E(G)$ such that $d_G(u) \le \lfloor \frac{\Delta}{2} \rfloor$ and $d_G(u) + d_G(v) \le \Delta + 1$. Since G is a critical edge Δ-choosable graph, all edges of $E(G - uv)$ can be colored from their lists of size Δ. There is at least $\Delta - (\Delta - 1) = 1$ color available from A_{uv} to color uv because $d_G(u) + d_G(v) \le \Delta + 1$. Thus, G is an edge Δ-choosable graph, a contradiction.

Let G be a critical totally $(\Delta+1)$-choosable graph and let $uv \in E(G)$ such that $d_G(u) \le \lfloor \frac{\Delta}{2} \rfloor$ and $d_G(u)+d_G(v) \le \Delta+1$. Since G a critical totally $(\Delta+1)$-choosable graph, all elements of $VE(G-uv)$ can be colored from their lists of size $\Delta+1$. Erase the color on u. Similar to the edge critical case, there is at least $(\Delta+1)-(\Delta-1)-1 = 1$ color available in A_{uv} to color uv. Now consider vertex u. Since $d_G(u) \le \lfloor \frac{\Delta}{2} \rfloor$, there is at least $(\Delta + 1) - (2 \times \lfloor \frac{n}{2} \rfloor) \ge 1$ colors available in A_u to color u. Thus, G is a totally Δ-choosable graph, a contradiction. □

A bipartite subgraph, denoted by F, of G is called a k-alternator of G with partite sets X, Y for some k $(2 \le k \le \lfloor \frac{\Delta}{2} \rfloor)$ such that $d_F(x) = d_G(x) \le k$ for each $x \in X$, and $d_F(y) \ge d_G(y) + k - \Delta$ for each $y \in Y$. Lemma 2.2 implies that $X \subseteq V(G)$ is an independent set of vertices since $d_G(x) \le \lfloor \frac{\Delta}{2} \rfloor$ for each $x \in X$.

Lemma 2.3. *There is no k-alternator F in G for any integer $k \in [2, \lfloor \frac{\Delta}{2} \rfloor]$.*

Proof. Suppose, to the contrary, there exits a k-alternator in G. Let F be a k-alternator of G with partite sets X, Y for some k $(2 \le k \le \lfloor \frac{\Delta}{2} \rfloor)$ such that $d_F(x) = d_G(x) \le k$ for each $x \in X$, and $d_F(y) \ge d_G(y) + k - \Delta$ for each $y \in Y$. Clearly, X is an independent set of vertices.

Let G be a critical edge Δ-choosable graph. It follows that we can color all edges in $G[V(G) - X]$ from their lists of size Δ. Now consider the edges between X and Y. Let xy be any edge of F where $x \in X$ and $y \in Y$ and A_{xy} be the list of xy. In the following we shall color the edges in F from their lists. Let A'_{xy} be the list of colors in A_{xy} that are still available to color xy in the list A_{xy} after the edges in $G[V(G) - X]$ were colored. It follows that xy has a list A'_{xy} of size at least $\max\{d_F(x), d_F(y)\}$, that is, $|A'_{xy}| \ge \Delta - (d_G(y) - d_F(y)) \ge d_F(y)$ and $|A'_{xy}| \ge \Delta - (d_G(y) - d_F(y)) \ge \Delta - d_G(y) + (d_G(y) + k - \Delta) \ge k \ge d_F(x)$. By Lemma 2.1, the edges of F can be colored from their lists. Thus, G is an edge Δ-choosable graph, a contradiction.

Let G be a critical totally $(\Delta+1)$-choosable graph. It follows that we can color all elements in $VE(G[V(G)-X])$ from their lists of size $\Delta + 1$. Now consider the edges between X and Y. Similarly, we can color all edges between X and Y because the size of each list for any edge xy between X and Y is increased by one and one color has been used on y, that is, $|A'_{xy}| \ge (\Delta+1) - (d_G(y) - d_F(y)) - 1 \ge d_F(y)$ and $|A'_{xy}| \ge (\Delta+1) - (d_G(y) - d_F(y)) - 1 \ge \Delta - d_G(y) + (d_G(y) + k - \Delta) \ge k \ge d_F(x)$. If we are coloring vertices, then each vertex $x \in X$ is now adjacent or incident to at most $2\lfloor \frac{\Delta}{2} \rfloor$ elements, and so there is at least one color available in A_x to color x. Thus, G is a totally $(\Delta + 1)$-choosable graph, a contradiction. □

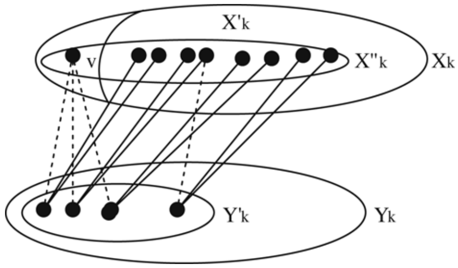

Fig. 1. Bipartite graph (X''_k, Y'_k) with $k = 3$.

Lemma 2.4. *For any integer $k \in [2, \lfloor\frac{\Delta}{2}\rfloor]$, let $X_k = \{x \in V(G) \mid d_G(x) \le k\}$ and $Y_k = \cup_{x\in X_k} N(x)$. If $X_k \ne \emptyset$, then there exists a bipartite subgraph M_k of G with partite sets X_k and Y_k such that $d_{M_k}(x) = 1$ for each $x \in X_k$ and $d_{M_k}(y) \le k - 1$ for each $y \in Y_k$.*

Proof. We will prove the existence of such a matching M_k by contradiction. Since this proof is constructed based on Lemma 2.3, that is, there is no k-alternator in either a critical edge Δ-choosable graph or a totally critical $(\Delta + 1)$-choosable graph, we do not have to distinguish two cases in the proof.

Let G be either a critical edge Δ-choosable graph or a totally critical $(\Delta + 1)$-choosable graph. By Lemma 2.3, X_k is an independent set of vertices. Let M'_k be a maximum bipartite subgraph with partite sets X'_k and Y_k, where $X'_k \subseteq X_k$, such that $d_{M'_k}(x) = 1$ for each $x \in X'_k$ and $d_{M'_k}(y) \le k - 1$ for each $y \in Y_k$. Note that there may be some isolated vertices under M'_k in Y_k. Since there is at least one edge from X_k to Y_k, M'_k is not empty.

In the following, we will show that $X'_k = X_k$. Suppose, to the contrary, $X'_k \subset X_k$. Let $v \in X_k \setminus X'_k$. A v-alternating path P_v in G is a path whose origin is v and edges are alternating between $E(G) \setminus E(M'_k)$ and $E(M'_k)$. We claim that if v' is a terminus of an v-alternating path and $v' \in Y_k$ then $d_{M'_k}(v') = k - 1$. Suppose there exist such a v-alternating path $P_v = vv_1v_2\cdots v_{2m+1}$ such that it terminates at $v_{2m+1} = v' \in Y_k$ and $d_{M'_k}(v_{2m+1}) < k - 1$. Then $M''_k = M'_k - \{v_1v_2, v_3v_4, \ldots, v_{2m-1}v_{2m}\} + \{vv_1, v_2v_3, \ldots, v_{2m}v_{2m+1}\}$ is another bipartite subgraph satisfying Lemma 2.4 and of size $|E(M_k)| + 1$, a contradiction to the maximality of M'_k. Let $Z = \{v_i \mid \text{where } v_0 = v, v_1, v_2, \ldots, v_m \text{ is a } v\text{-alternating path}\}$. Set $X''_k = Z \cap X_k$ and $Y'_k = Z \cap Y_k$ (see Fig. 1). Then $X''_k = \{v\} \cup (Z \cap X'_k)$ and $\cup_{x\in X''_k} N(x) = Y'_k$. If there is a vertex $x \in X''_k$ such that it has a neighbor $y \notin Y'_k$, we can obtain a longer v-alternating path from v to y passing x. Hence $\cup_{x\in X''_k} N(x) \subseteq Y'_k$. Since $Y'_k = Z \cap Y_k$, there is a v-alternating path containing y for each vertex $y \in Y'_k$. This implies that there exists a vertex $x \in X''_k$ such that y is a neighbor of x. Thus, $Y'_k \subseteq \cup_{x\in X''_k} N(x)$. It follows that $\cup_{x\in X''_k} N(x) = Y'_k$. Let M''_k be the induced bipartite subgraph of G with bipartitions X''_k, Y'_k. In the following we show that M''_k is a k-alternator. By the definition of X_k, $d_{M''_k}(x) = d_G(x) \le k$ for any $x \in X''_k$. Clearly, $d_{M'_k}(y) = k - 1$ where $y \in Y'_k$. Since each vertex $y \in Y'_k$ can be reached by a v-alternating path, there must be at least one edge e incident with y such that $e \in E(M''_K) \setminus E(M'_k)$. So $d_{M''_k}(y) \ge d_{M'_k}(y) + 1 = k - 1 + 1 \ge k + d_G(y) - \Delta$ for each $y \in Y'_k$. Hence M''_k is a k-alternator of G, a contradiction with Lemma 2.3. This completes the proof. $\square$

In Lemma 2.4 we show that for any given integer $k \in [2, \lfloor\frac{\Delta}{2}\rfloor]$ there is a many to one matching M_k from Y_k to X_k in G if G is either a critical edge Δ-choosable or a totally critical $(\Delta + 1)$-choosable graph. We call y the *k-master* of x if $xy \in M_k$ and $x \in X_k$. So every i-vertex has a j-master where $2 \le i \le \lfloor\frac{\Delta}{2}\rfloor$ and $j = i, i + 1, \ldots, \lfloor\frac{\Delta}{2}\rfloor$. Lemma 2.4 will be used to define new charge functions in the next section. Note that G is a general critical graph, i.e., it need not be graphs embedded on hyperbolic surfaces, and there is no restriction on the maximum degree of G in Lemma 2.4.

3. Main results

Proof of Theorem 1.4. We shall complete the proof of the theorem by using discharging in order to obtain a contradiction. Let G be either a critical edge Δ-choosable graph or a totally critical $(\Delta + 1)$-choosable graph. By Lemma 2.4, every i-vertex has a j-master in G where $2 \le i \le \lfloor\frac{\Delta}{2}\rfloor$ and $j = i, i + 1, \ldots, \lfloor\frac{\Delta}{2}\rfloor$. We shall use this to define discharging rules. In the following, we use only the discharging method, and, in turn, the restrictions on the maximum degree and embeddability of G remain unchanged.

By Euler's formula $|V| - |E| + |F| = \varepsilon$, we have $|E| \le 3(|V| - \varepsilon)$, that is,

$$S = \sum_{v \in V} (d_G(v) - 6) \le -6\varepsilon.$$

We first define a charge, $w(v)$, of G. Let $w(v) = d_G(v) - 6$ for each $v \in V(G)$. We now redistribute the initial charge $w(v)$ and form a new charge $w^*(v)$. The discharging rule is as follows:

Each i-vertex receives 1 from all its j-masters, where $2 \le i \le 5$ and $j = i, \ldots, 5$.

Let v be any vertex in $V(G)$. If $2 \le d_G(v) \le 5$, then $w^*(v) = 0$ since it receives $6 - d_G(v)$ from each of its j-masters where $j = d_G(v), d_G(v) + 1, \ldots, 5$.

Suppose $6 \le d_G(v) \le \Delta - 4$. Note that $\lfloor \frac{\Delta}{2} \rfloor \ge \lfloor \frac{10+7}{2} \rfloor = 8$. For any $u \in N(v)$, if $d_G(u) \le 5$ then $d_G(u) \le \lfloor \frac{10+7}{2} \rfloor = 8$, and, in turn, $d_G(v) + d_G(u) < \Delta + 2$, a contradiction with Lemma 2.2. Hence, $d_G(u) \ge 6$ for any vertex $u \in N(v)$. It follows that v will neither receive any charge nor give any charge through the discharge. So $w^*(v) = w(v) \ge 0$ for $6 \le d_G(v) \le \Delta - 4$.

If $d_G(v) = \Delta - 3$, then $d_G(u) \ge 5$ for any $u \in N(v)$. This implies that v may be a 5-master of at most four vertices in G. So $w^*(v) \ge w(v) - 4 = ((\Delta - 3) - 6) - 4 = \Delta - 13$. If $d_G(v) = \Delta - 2$, then $d_G(u) \ge 4$ for any $u \in N(v)$, and it may be a 5-master of at most four vertices and a 4-master of at most three vertices. So $w^*(v) \ge w(v) - 4 - 3 = \Delta - 15$. Similarly, $w^*(v) \ge w(v) - 4 - 3 - 2 = \Delta - 16$ if $d_G(v) = \Delta - 1$, and $w^*(v) \ge w(v) - 4 - 3 - 2 - 1 = \Delta - 16$ if $d_G(v) = \Delta$. Since $\Delta \ge \sqrt{25 - 24\varepsilon} + 10 \ge 17$, $w^*(v) \ge \Delta - 16 > 0$ if $d_G(v) \ge \Delta - 3$.

Let $X = \{x \in V(G) \mid d_G(x) \le \lfloor \frac{\Delta}{2} \rfloor\}$. By Lemma 2.2, X is an independent set of vertices. Suppose $|V(G - X)| \le \lfloor \frac{\Delta}{2} \rfloor + 1$. Then $d_{G-X}(v) \le \lfloor \frac{\Delta}{2} \rfloor$ for any $v \in V(G) - X$. Let $Y = \cup_{x \in X} N(x)$ and F be the induced bipartite subgraph with partite sets X, Y. Then for any vertex $y \in Y$, we have $d_F(y) = d_G(y) - d_{G-X}(y) \ge d_G(y) - \lfloor \frac{\Delta}{2} \rfloor \ge \lfloor \frac{\Delta}{2} \rfloor + d_G(y) - \Delta$, that is, F is a $\lfloor \frac{\Delta}{2} \rfloor$-alternators of G, a contradiction. So $|V(G - X)| \ge \lfloor \frac{\Delta}{2} \rfloor + 2$. Since $\Delta \ge \lceil \sqrt{25 - 24\varepsilon} \rceil + 10$, $\Delta - 9 \ge \lfloor \frac{\Delta}{2} \rfloor$. So $w^*(v) \ge \lfloor \frac{\Delta}{2} \rfloor - 7$ if $d_G(v) \ge \lfloor \frac{\Delta}{2} \rfloor + 1$. Thus $S = \sum_{v \in V(G)} w(v) = \sum_{v \in V(G)} w^*(v) \ge (\lfloor \frac{\Delta}{2} \rfloor + 2)(\lfloor \frac{\Delta}{2} \rfloor - 7) \ge \frac{1}{4}(\sqrt{25 - 24\varepsilon} + 10 - 1 + 4)(\sqrt{25 - 24\varepsilon} + 10 - 1 - 14) > -6\varepsilon$, a contradiction. This completes the proof. □

The concept of k-alternator was first introduced by Borodin, Kostochka and Woodall in [1]. Our definition of k-altenator is similar to their definition. We further extended this concept in Lemma 2.4 and used it to prove Theorem 1.4. It is easy to check that

$$\lim_{\varepsilon \to \infty} \frac{\sqrt{25 - 24\varepsilon} + 10}{H(\varepsilon)} = 2.$$

That is, there is a gap between the Heawood number (see Table 1) and the lower bound of Δ in Theorem 1.4. There may be room for a further improvement on the lower bound of the maximum degree in Theorem 1.4. We conclude this paper with another theorem based on Lemma 2.4, which has smaller lower bound of Δ but with an additional condition.

Theorem 3.1. *Let G be a graph embeddable on a surface of Euler characteristic $\varepsilon \ge 0$ with maximum degree Δ. Then $\chi''_{\text{list}}(G) = \Delta + 1$ in each of the following cases:*

(i) $\Delta \ge 9$ *and no two triangles have a common edge;*
(ii) $\Delta \ge 8$ *and no two triangles have a common vertex.*

Proof. Let G be a critical totally $(\Delta + 1)$-choosable graph. By Lemma 2.4, every i-vertex has a j-master in G where $2 \le i \le \lfloor \frac{\Delta}{2} \rfloor$ and $j = i, i + 1, \ldots, \lfloor \frac{\Delta}{2} \rfloor$. We shall use this to define discharging rules. In the following, we use only the discharging method and, in turn, the restrictions on the maximum degree and embeddability of G remain unchanged.

By Euler's formula $|V| - |E| + |F| = \varepsilon$, we have that

$$S_3 = \sum_{v \in V} (d_G(v) - 4) + \sum_{f \in F} (r(f) - 4) = -4(|V| - |E| + |F|) = -4\varepsilon \le 0.$$

Let

$$w(x) = \begin{cases} d_G(x) - 4, & \text{if } x \in V, \\ r(x) - 4, & \text{if } x \in F. \end{cases}$$

We now redistribute the initial charge $w(x)$ and form a new charge $w^*(x)$. Recall that in Lemma 2.4 we showed that for each i-vertex where $i = 2, 3$ there is an i-master. The discharging rules are as follows:

R(d)1: Each 2-vertex receives 1 from its 2-master and 1 from its 3-master;
R(d)2: Each 3-vertex receives 1 from its 3-master;
R(d)3: Each 3-face receives $\frac{1}{2}$ from its incident vertex v of degree at least 5.

If a vertex v has degree less than 5, then any vertex u adjacent to v must be of degree at least $\lfloor \frac{\Delta}{2} \rfloor = 5$ since $d_G(u) + d_G(v) \geq \Delta + 2 \geq 11$. It follows that each 3-face f is incident with at least two vertices of degree at least 5, and, in turn, it implies that $w^*(f) \geq 0$. Clearly, $w^*(f) \geq 0$ if $r(f) > 3$. For any vertex v, $w^*(v) = 0$ if $d_G(v) = 2, 3, 4$. For any vertex v of degree 5, 6 or 7, it is only adjacent to the vertices of degree at least 5. This implies that v cannot be i-master of any other vertices where $i = 2, 3$. It follows that it may give $\frac{1}{2}$ to at most $\lfloor \frac{d_G(v)}{2} \rfloor$ triangles since no two triangles have a common edge. Thus, $w^*(v) \geq w(v) - \lfloor \frac{d_G(v)}{2} \rfloor \times \frac{1}{2} \geq 0$. Similarly, any vertex v of degree 8 may be a 3-master of two other vertices and can belong to at most four triangles. It follows that $w^*(v) \geq w(v) - 2 - 4 \times \frac{1}{2} = 0$.

For any vertex v of degree at least 9, it may be a 3-master of two other vertices, a 2-master of another vertex, and can belong to at most $\lfloor \frac{d_G(v)}{2} \rfloor$ triangles. Hence, $w^*(v) \geq w(v) - 2 - 1 - \lfloor \frac{d_G(v)}{2} \rfloor \times \frac{1}{2} \geq 0$ if $d_G(v) \geq 9$. Clearly, $w^*(v) > 0$ for $d_G(v) \geq 10$, a contradiction with $S_3 \leq 0$. In the following, we assume that $d_G(v) = 9$. By Lemma 2.2, any 2-vertex can be only adjacent to the vertices with maximum degree. Furthermore, Lemma 2.4 implies that the number of vertices of degree 2 is less than or equal to the number of vertices of degree Δ. Since there is no 2-alternator in G, the number of vertices of degree 2 is strictly less than the number of vertices of degree Δ. It follows that there exists at least one vertex, v, of degree Δ, that is not a 2-master of any vertex if $\Delta = 9$. If $\Delta \geq 10$, no vertex with $d_G(v) = 9$ can be a 2-master. In both the cases, there is a vertex v such that $w^*(v) \geq w(v) - 2 - \lfloor \frac{d_G(v)}{2} \rfloor \times \frac{1}{2} > 0$. Therefore, $S_3 = \sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w^*(x) > 0$, a contradiction with $S_3 \leq 0$. This proves (i).

We can use the same new charge function $w^*(x)$ as of (i) and similarly prove (ii). $\square$

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